

# On the wake generated by a planet in a disc

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## ABSTRACT

A planet of low mass orbiting in a two-dimensional gaseous disc generates a one-armed spiral wake. We explain this phenomenon as the result of constructive interference between wave modes in the disc, somewhat similar to the Kelvin wedge produced in the wake of a ship. The same feature is not expected in a three-dimensional disc with thermal stratification.

**Key words:** accretion, accretion discs – hydrodynamics – planets and satellites: general – waves.

## 1 INTRODUCTION

The dynamical interaction between a planet and the disc in which it forms has important consequences for the evolution of the orbital elements of the planet. In particular, any imbalance between the torques exerted on the planet by the parts of the disc interior and exterior to its orbit causes the planet to migrate radially through the disc (e.g. Ward 1986). The planet–disc interaction has been studied using linear theory and, more recently, non-linear numerical simulations. Most of this work uses a two-dimensional description that ignores the vertical structure and motion of the disc.

In the linear theory (Goldreich & Tremaine 1979) the planetary potential is analysed into Fourier components, each having a definite azimuthal wavenumber and angular pattern speed. For each Fourier mode, the response of the disc is determined by an inhomogeneous linear wave equation that includes resonances at various radii. Angular momentum is exchanged between the planet and the disc mainly in the vicinity of Lindblad resonances (at which waves are launched) and the corotation resonance (where angular momentum is deposited). Goldreich & Tremaine (1979) provided analytical formulae for these torques, while Korycansky & Pollack (1993) determined them by solving the linear wave equations numerically.

Neither of these studies was concerned with the form of the disturbance generated in the disc, which is obscured in the Fourier representation. In contrast, numerical simulations of the planet–disc interaction (e.g. Bryden et al. 1999; Kley 1999; Lubow, Seibert & Artymowicz 1999) aim to solve the non-linear fluid dynamical equations in real space. Most such simulations have treated the case of a Jovian planet, for which the planet–disc interaction is highly non-linear, especially close to the planet where a gap is opened in the disc. Further away, spiral shock waves are distinctly seen.

A detailed comparison between linear and non-linear theory has been made by Miyoshi et al. (1999), who compared the torques generated in a local, three-dimensional simulation (an isothermal, shearing-sheet model) with those derived by solving the linear wave equations. They did not compare the form of the disturbance in real space, however.

For planets of lower mass, a gap is not opened and the response of the disc should be well described by linear theory. Simulations for a terrestrial planet (Artymowicz 2001) indicate that a remarkable one-armed spiral wake is formed, but there is no evidence for features associated with individual Lindblad resonances. The relation between the wake and the linear Fourier analysis has been obscure. In this letter we explain this relation by showing how the wake is formed from the superposition of individual Fourier modes. We also discuss the differences that are to be expected in future three-dimensional simulations.

## 2 LINEAR HYDRODYNAMIC WAVES IN A TWO-DIMENSIONAL DISC

### 2.1 Dispersion relation and pitch angle

Consider a two-dimensional gaseous disc around a star of mass  $M$ . Let the disc have angular velocity  $\Omega(r)$ , surface density  $\Sigma(r)$  and adiabatic sound speed  $v_s(r)$ . Suppose that a planet of mass  $M_p$  orbits within the disc on a circular orbit of radius  $r_p$  and angular velocity  $\Omega_p$ . The planet interacts gravitationally with the disc and excites hydrodynamic waves having angular pattern speed  $\Omega_p$ . We assume that the waves may be described using linear theory.

For linear waves, any wave quantity  $X$  can be written in the form

$$X(r, \phi, t) = \text{Re} \left\{ \tilde{X}(r) \exp[i\Phi_m(r, \phi, t)] \right\}, \quad (1)$$

where  $\tilde{X}$  is an amplitude that varies slowly with  $r$ , while

$$\Phi_m = \int k(r) dr + m(\phi - \Omega_p t) \quad (2)$$

is a phase that varies rapidly with  $r$ . Here  $k$  is the radial wavenumber, which is real in regions of space where the wave propagates, and  $m$  is the azimuthal wavenumber. We consider only non-axisymmetric waves, and take  $m$  to be a positive integer by convention. We initially leave the constant of integration in equation (2) indefinite.

The dispersion relation for tightly wound hydrodynamic waves in a two-dimensional gaseous disc is

$$[m(\Omega - \Omega_p)]^2 = \kappa^2 + v_s^2 k^2, \quad (3)$$

where  $\kappa(r)$  is the epicyclic frequency. We neglect the effects of self-gravitation and dissipative processes. The wavefronts, or lines of constant phase, are spirals defined by the equation

$$\frac{d\phi}{dr} = -\frac{k}{m}. \quad (4)$$

We note that, if the term  $\kappa^2$  in equation (3) may be neglected compared to the other terms, the pitch angle  $\arctan(m/\kappa r)$  is independent of  $m$ . This raises the possibility, in principle, that constructive interference may occur between waves of different azimuthal wavenumber. Since  $\kappa$  and  $\Omega$  are usually of comparable magnitude (indeed, they are equal for a Keplerian disc), the term  $\kappa^2$  is relatively small when  $m$  is large, except close to the corotation radius where  $\Omega = \Omega_p$ .

## 2.2 Disc model

We assume that the disc is Keplerian, so that<sup>1</sup>

$$\kappa = \Omega = \left(\frac{GM}{r^3}\right)^{1/2}. \quad (5)$$

The corotation radius is then

$$r_c = \left(\frac{GM}{\Omega_p^2}\right)^{1/3}. \quad (6)$$

We assume further that the sound speed is given by

$$v_s = \epsilon \left(\frac{GM}{r}\right)^{1/2}, \quad (7)$$

where  $\epsilon$  is a constant. This convenient assumption, which is not critical to the analysis that follows, implies that the disc has a constant angular semi-thickness  $H/r$  proportional to  $\epsilon$ , and a constant Mach number  $\epsilon^{-1}$ .

Adopting units such that  $GM = \Omega_p = r_c = 1$ , we find

$$k^2 = \frac{m^2}{\epsilon^2 r^2} \left(r^{3/2} - r_+^{3/2}\right) \left(r^{3/2} - r_-^{3/2}\right), \quad (8)$$

where

$$r_{\pm} = \left(1 \pm \frac{1}{m}\right)^{2/3} \quad (9)$$

are the radii of the outer and inner Lindblad resonances for mode  $m$ . Waves are launched by the planet at the Lindblad resonances and propagate into  $r > r_+$  (with  $k > 0$ ) and into  $r < r_-$  (also with  $k > 0$ ). The sign of  $k$  is chosen such that the radial group velocity is directed away from the planet.

<sup>1</sup> Although torque calculations are sensitive to slight deviations from Keplerian rotation, e.g. resulting from radial pressure gradients, they are not important in the present analysis.

## 2.3 Waves in the outer disc

Consider the waves launched in the outer disc, i.e. the disc exterior to the planet's orbit. The precise phase of mode  $m$  for  $r > r_+$  is given by

$$\Phi_m = \frac{\pi}{4} + \int_{r_+}^r k(r') dr' + m(\phi - t), \quad (10)$$

where the term  $\pi/4$  is the phase shift associated with the resonance. This can be deduced from the behaviour of the Airy function, which describes the launched wave near the Lindblad resonance.

We first obtain an *estimate* of the phase by a method equivalent to neglecting the term  $\kappa^2$  in the dispersion relation. As noted above, this is appropriate when  $m$  is large. Approximating  $r_{\pm} \approx 1$ , we have

$$\int k(r) dr \approx \frac{m}{\epsilon} \int r^{-1} (r^{3/2} - 1) dr, \quad (11)$$

and therefore

$$\Phi_m \approx \frac{\pi}{4} + \frac{2m}{3\epsilon} \left(r^{3/2} - \frac{3}{2} \ln r - 1\right) + m(\phi - t). \quad (12)$$

Constructive interference occurs near the curve  $\phi = \varphi(r, t)$  defined by

$$\varphi = t - \frac{2}{3\epsilon} \left(r^{3/2} - \frac{3}{2} \ln r - 1\right), \quad (13)$$

since we have

$$\Phi_m \approx \frac{\pi}{4} + m(\phi - \varphi). \quad (14)$$

We can, however, calculate the phase exactly by changing to the variable  $x = r^{3/2}$  and making use of the indefinite integral

$$\begin{aligned} \int x^{-1} (x-a)^{1/2} (x-b)^{1/2} dx &= (x-a)^{1/2} (x-b)^{1/2} \\ &\quad - (a+b) \ln [(x-a)^{1/2} + (x-b)^{1/2}] - (ab)^{1/2} \\ &\quad \times \ln \left[ \frac{(a+b)x - 2ab - 2(ab)^{1/2}(x-a)^{1/2}(x-b)^{1/2}}{x} \right]. \end{aligned} \quad (15)$$

We write the exact solution as

$$\Phi_m = \frac{\pi}{4} + m(\phi - \varphi) - \frac{2}{3\epsilon} \Delta_m(r). \quad (16)$$

The last term measures the error in the approximate method, and will determine whether constructive interference really does occur on the curve  $\phi = \varphi$ . The approximate method always slightly overestimates both the wavenumber  $k$  and the range of integration in the phase integral. It follows that the residue  $\Delta_m$  has the properties

$$\Delta_m > 0, \quad \frac{d\Delta_m}{dr} > 0. \quad (17)$$

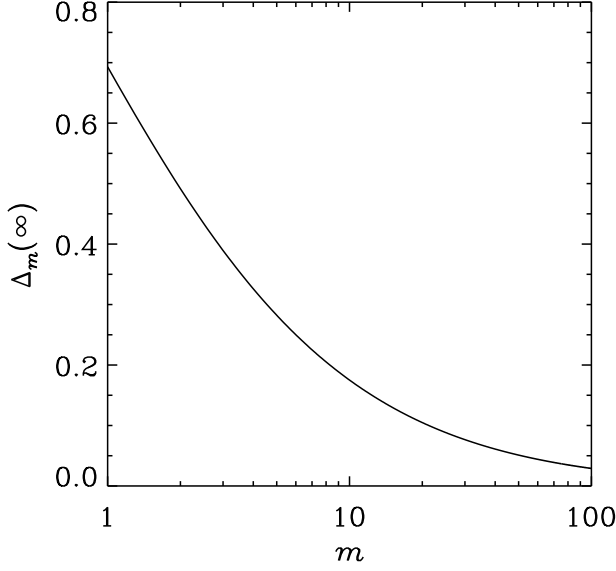
The expansion for large  $r$  is

$$\Delta_m(r) = \Delta_m(\infty) - \frac{1}{2mr^{3/2}} - \frac{1}{4mr^3} + O(r^{-9/2}), \quad (18)$$

where (see Fig. 1)

$$\Delta_m(\infty) = m \ln(2m) + (m^2 - 1)^{1/2} \ln [m - (m^2 - 1)^{1/2}]. \quad (19)$$

A further property is that, for any fixed  $r$ ,



**Figure 1.** The residue  $\Delta_m(\infty)$ .

$$\lim_{m \rightarrow \infty} \Delta_m = 0. \quad (20)$$

In fact, for large  $m$ ,

$$\Delta_m \sim \frac{1}{2m} \ln \left[ 2e^{1/2} m (1 - r^{-3/2}) \right]. \quad (21)$$

Waves of different  $m$  add constructively on the curve  $\phi = \varphi$  provided that their phases lie within a range less than approximately  $\pi$ . This certainly occurs for waves of sufficiently large  $m$ , because of property (20). Constructive interference may fail for low values of  $m$  for which  $\Delta_m \gtrsim 3\pi\epsilon/2$ , since such modes are out of phase with waves of high  $m$ . The properties (17) ensure that the limit  $r \rightarrow \infty$  is the worst case. The residue is plotted in Fig. 2 for  $m = 1, \dots, 10$ . For  $\epsilon = 0.1$ , typical of protoplanetary discs, constructive interference fails only for  $1 \leq m \leq 2$  in the limit of large  $r$ .

Provided that a range of azimuthal wavenumbers is present, and there is no special selection rule (e.g. one that selects azimuthal wavenumbers that are all multiples of 2), there is no other curve on which constructive interference occurs consistently. The result is a *one-armed spiral wake* following the curve  $\phi = \varphi$ .

The disturbance generated by a planet orbiting in the disc is dominated by azimuthal wavenumbers  $m \approx m_* \approx 1/(2\epsilon)$  (Goldreich & Tremaine 1980). Accordingly, the azimuthal thickness of the wake is  $\delta\phi \approx 2\pi/m_* \approx 4\pi\epsilon$ . The radial thickness is smaller,  $\delta r/r \approx \epsilon \delta\phi \approx 4\pi\epsilon^2$ .

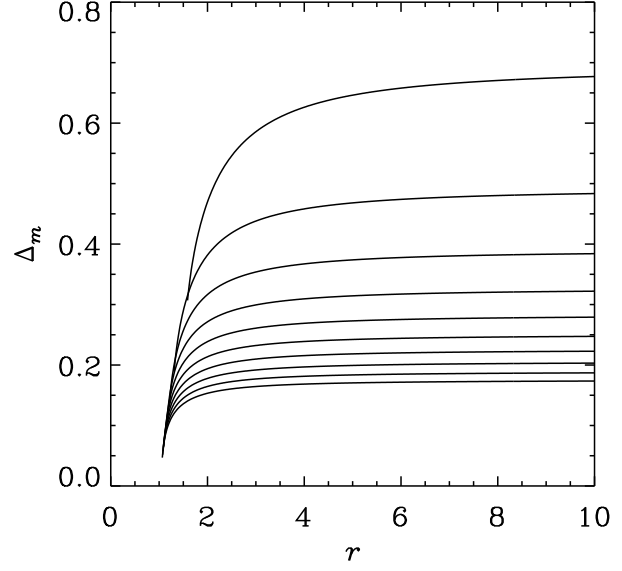
#### 2.4 Waves in the inner disc

Waves launched in the inner disc behave very similarly. The precise phase of mode  $m$  for  $r < r_-$  is given by

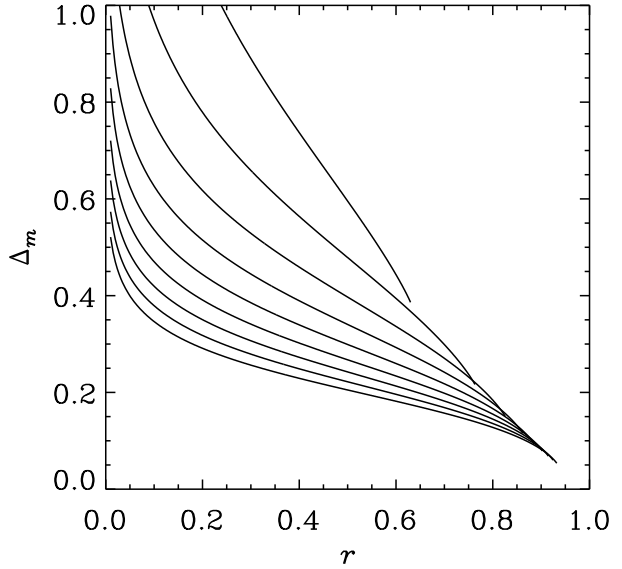
$$\Phi_m = \frac{\pi}{4} + \int_{r_-}^r k(r') dr' + m(\phi - t), \quad (22)$$

where, again,  $k > 0$ . We again write the exact solution in the form

$$\Phi_m = \frac{\pi}{4} + m(\phi - \varphi) + \frac{2}{3\epsilon} \Delta_m(r), \quad (23)$$



**Figure 2.** The residue  $\Delta_m$  in the outer disc. Modes  $m = 1, \dots, 10$  are shown from top to bottom.



**Figure 3.** The residue  $\Delta_m$  in the inner disc. Modes  $m = 1, \dots, 10$  are shown from top to bottom.

where now

$$\varphi = t + \frac{2}{3\epsilon} \left( r^{3/2} - \frac{3}{2} \ln r - 1 \right). \quad (24)$$

In the inner disc the residue has the properties

$$\Delta_m > 0, \quad \frac{d\Delta_m}{dr} < 0. \quad (25)$$

The limiting form for small  $r$  is

$$\Delta_m(r) = -\frac{3}{2} [m - (m^2 - 1)^{1/2}] \ln r + O(1). \quad (26)$$

Constructive interference does *eventually* fail for all  $m$  in the limit  $r \rightarrow 0$ .

The residue is plotted in Fig. 3 for  $m = 1, \dots, 10$ . The limiting form for large  $m$  is

$$\Delta_m \sim \frac{1}{2m} \ln [2e^{1/2} m (r^{-3/2} - 1)]. \quad (27)$$

### 3 NUMERICAL CALCULATION

To verify the hypothesis that the wake is formed through the constructive interference of Fourier modes, we have calculated the linear response of the disc numerically using methods similar to Korycansky & Pollack (1993).<sup>2</sup> We adopted a two-dimensional barotropic disc model equivalent to that described in Section 2.2, and having a surface density  $\Sigma \propto r^{-3/2}$ . Outgoing-wave boundary conditions were applied at  $r = 0.3$  and  $r = 3$ . The potential of the planet was smoothed to simulate the non-zero vertical extent of the disc, with a smoothing length  $\epsilon r_p$  comparable to the semi-thickness  $H$ . The linear solutions for modes  $m = 1$  to  $m = 100$  were calculated and synthesized in real space.

The results are shown in Figs 4 and 5, where we compare the predicted shape of the wake given by equations (13) and (24) with the numerical calculation. The agreement is excellent. The numerical calculation shows that the wake has some non-trivial internal structure, often consisting of both a trough and a larger peak. These details are likely to depend to some extent on the disc model.

Although the waves are generated at Lindblad resonances, individual resonances cannot be observed in the complete solution because they overlap. The wake is generated in a perfectly smooth manner. It is clearly established after the synthesis of the first 10 modes in the case of  $\epsilon = 0.05$ , or the first 5 modes in the case of  $\epsilon = 0.1$ . The addition of higher-order modes merely increases the finesse of the interference pattern.

### 4 DISCUSSION

We have considered the dynamical interaction between a planet of low mass and a two-dimensional gaseous disc in which it orbits. Although the planet generates disturbances of all azimuthal wavenumbers, the wave modes interfere constructively on a unique curve and form a one-armed spiral wake. The waveform based on linear theory (see Figs 4 and 5) is in good agreement with that found in non-linear planet-disc simulations (e.g. Artymowicz 2001).

The formation of a coherent structure by constructive interference is reminiscent of the Kelvin wedge produced in the wake of a ship. In the present case the effect is more complete, as almost all modes interfere constructively, rather than a band of wavenumbers.

Recently, Goodman & Rafikov (2001) calculated the linear wake produced by a terrestrial planet in a local, shearing-sheet model of a two-dimensional disc. As they noted, the formation of the wake enhances the amplitude of the disturbance in a thin structure and therefore increases the likelihood of non-linear effects such as shock formation.

The precise relation between a linear wake and the spiral shocks observed in numerical simulations involving planets of larger mass (e.g. Lubow et al. 1999) is not entirely clear, however. In such simulations a dominant one-armed shock is formed, which follows a similar curve to the linear prediction, but other features are also present. The flow near the planet is also different in the non-linear regime.

Although most treatments of the planet-disc interaction have used a two-dimensional description of the disc, this is in fact difficult to justify. First, the wave modes in a three-dimensional disc are, in general, different from those in a two-dimensional disc. If the disc is vertically isothermal, there does exist a two-dimensional mode with the same dispersion relation as equation (3) (Lubow & Pringle 1993). In that case, a wake would be formed in much the same way. In a vertically thermally stratified disc, however, the waves generated by the planet have a quite different dispersion relation. For example, in a polytropic model, which represents a highly optically thick disc with vertically distributed energy dissipation, the equivalent mode behaves like a surface gravity wave having an approximate dispersion relation

$$[m(\Omega - \Omega_p)]^2 \approx gk \quad (28)$$

sufficiently far from the planet, where  $g = \Omega^2 H$  is the vertical gravitational acceleration at the surface of the disc (Ogilvie 1998; Lubow & Ogilvie 1998). Spiral waves with different values of  $m$  then have different pitch angles and there is no possibility of consistent constructive interference on a curve.<sup>3</sup> We therefore anticipate that the disturbance generated by a planet in a thermally stratified disc will have a quite different structure.

In addition, the flow near the planet is likely to be three-dimensional if the radius of the planet's Roche lobe is comparable to or less than the semi-thickness of the disc. Therefore, although this analysis may explain some features observed in numerical simulations, the realities of the planet-disc interaction are likely to be more complicated.

We remark that an accurate linear calculation of the planetary wake, and of the associated migration rate, in a thermally stratified, three-dimensional disc would be a demanding numerical problem, and has not been carried out.

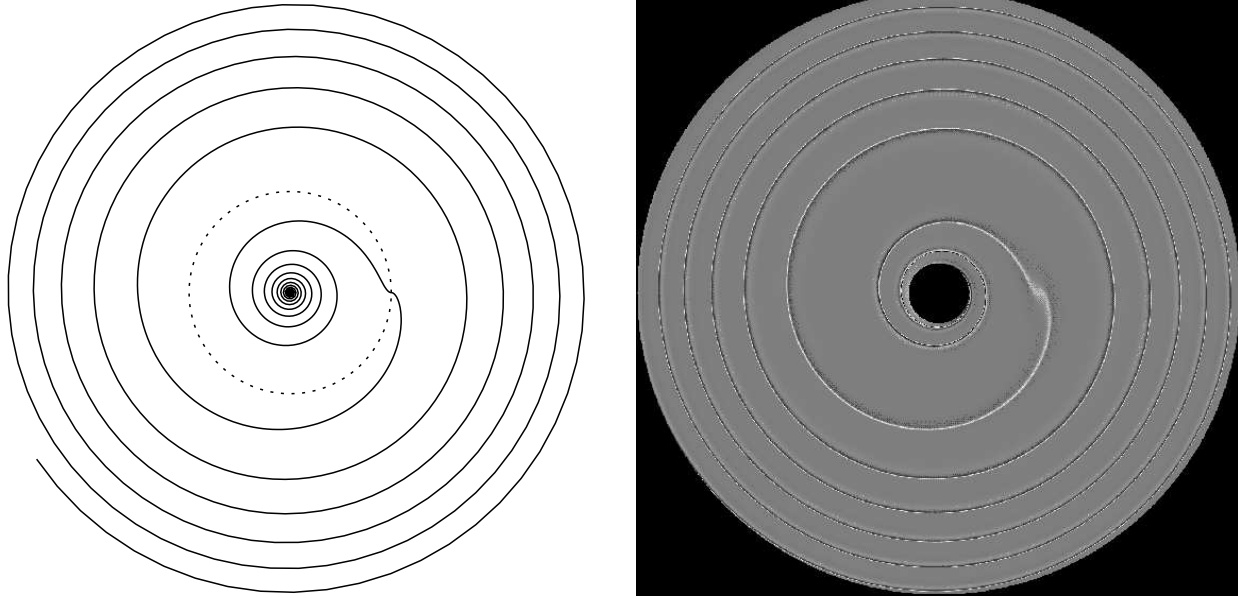
In summary, the one-armed wake is a consequence of having a spectrum of two-dimensional waves with different azimuthal wavenumbers  $m$  and approximately matching phases. Any mechanism capable of producing such waves will result in a similar one-armed wake. We have shown that resonantly launched waves in a two-dimensional gaseous disc naturally meet this criterion.

### ACKNOWLEDGMENTS

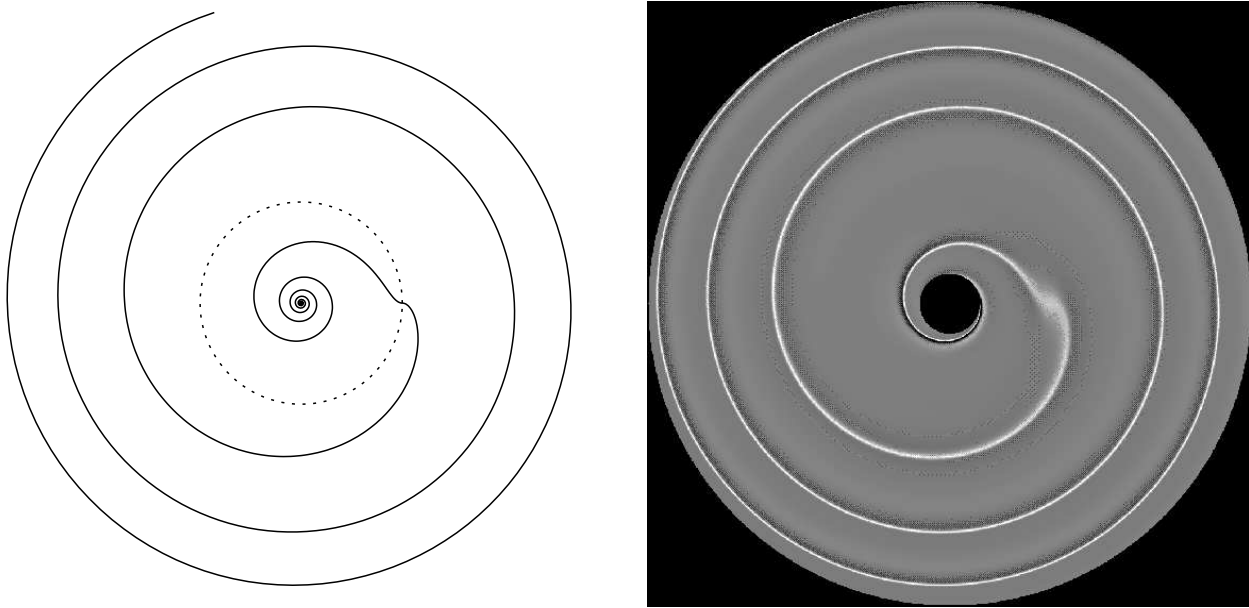
We are grateful to Marcus Roper for carrying out an early version of the calculation in Section 3. GIO acknowledges the support of the Royal Society through a University Research Fellowship. SHL acknowledges support from the Institute of Astronomy visitor programme and from NASA grants NAG5-4310 and NAG5-10732.

<sup>2</sup> We caution the reader of that paper of a number of errors, especially on p. 164.

<sup>3</sup> Although surface gravity waves are precisely what is involved in a ship wake, the analogy with the Kelvin wedge breaks down when the shear in the disc is taken into account.



**Figure 4.** Left: Predicted shape of the spiral wake for  $\epsilon = 0.05$ , based on equations (13) and (24). The dotted line represents the corotation circle,  $r = 1$ . The planet is located at  $(1, 0)$ , and the outer radius plotted is  $r = 3$ . Right: Numerically calculated spiral wake for  $\epsilon = 0.05$ . The enthalpy perturbation is plotted using a linear grey-scale from negative (black) to positive (white). The maximum intensity corresponds to a fractional surface density perturbation, at  $r = 1$ , of  $10^4 (M_p/M)$ .



**Figure 5.** As for Fig. 4, but with  $\epsilon = 0.1$ . The scale of the perturbation is  $10^3 (M_p/M)$ .

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